Spatial competition on 2-dimensional markets and networks when consumers don't always go to the closest firm

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Abstract. We investigate the strategic behavior of firms in a Hotelling spatial setting. The innovation is to combine two important features that are ubiquitous in real markets: (i) the location space is two-dimensional, often with physical restrictions on where firms can locate; (ii) consumers with some probability shop at firms other than the nearest. We characterise convergent Nash equilibria (CNE), in which all firms cluster at one point, for several alternative markets. In the benchmark case of a square convex market, we provide a new direct geometric proof of a result by Cox (1987) that CNE can arise in a sufficiently central part of the market. The convexity of the square space is of restricted realism, however, and we proceed to investigate networks, which more faithfully represent a stylised city's streets. In the case of a grid, we characterise CNE, which exhibit several new phenomena. CNE in more central locations tend to be easier to support, echoing the unrestricted square case. However, CNE on the interior of edges differ substantially from CNE at nodes and follow quite surprising patterns. Our results also highlight the role of positive masses of indifferent consumers, which arise naturally in a network setting. In most previous models, in contrast, such masses cannot exist or are assumed away as unrealistic.

Keywords: Spatial competition · Network · Grid · Nash equilibrium.

1 Introduction

The classical Hotelling (1929) model of spatial competition can be generalised in several directions. Hotelling assumed (and many researchers after him) that both firms and consumers are located on the unit interval $[0, 1]$, representing "Main Street". In the simplest model, it is assumed that customers patronise the closest retailer and the firms set an identical mill price of the good they provide, hence firms compete only by choosing their locations. Hotelling showed that for two competing firms the only equilibrium that exists is convergent and both firms locate at the midpoint of the interval. For three firms, no equilibria can exist, and for four or more firms the equilibria are characterised by Eaton and Lipsey (1975)—they are non-convergent with candidates occupying a rather diverse set of positions.

One important direction of generalisation was to change the nature of the space of possible locations. Salop (1979) considered, rather than a line, a circle. Eaton and Lipsey (1975) believed that, if the number of firms is more than two, Nash equilibria on a bounded two-dimensional location space are unlikely to exist. Okabe and Suzuki (1987) investigated this conclusion further computationally and Aoyagi and Okabe (1993) studied how the shape of the two-dimensional market affects the existence of an equilibrium.

On reflection, a two-dimensional location space also lacks realism. Indeed, the map of any city shows that it is structured as a network, with both firms and customers located along the city streets. In Sarkar et al. (1997), a network represents spatially separated markets with consumers located at nodes only. Pálvölgyi (2011) and Fournier and Scarsini (2019) consider the classical location game on a network (where customers and firms may locate anywhere)—when the number of firms is large enough, a pure strategy non-convergent Nash equilibrium exists and they show how to construct it. Fournier (2019) considers approximate equilibria under more general consumer distributions on the network.

In all the above work it is assumed that each consumer buys from the closest store. The second feature we incorporate into our model is the relaxation of this assumption. This approach was pioneered by Cox (1987; 1990) and further investigated by Cahan and Slinko (2017; 2018) and Cahan et. al (2018). Cox suggested that, while a consumer usually shops at the closest firm, there is some probability of shopping at more distant firms. Let the number of firms be m and $\mathbf{p} = (p_1, \ldots, p_m)$ be a vector of probabilities, where p_i is the probability that a consumer buys from the ith most distant firm, $p_1 \geq p_2 \geq \cdots \geq p_m$, and $p_1 > p_m$.[4] The classical Hotelling model and most of the previous literature (including, to our knowledge, all papers on networks) then corresponds to the special case where the probability vector is given by $\mathbf{p} = (1, 0, \ldots, 0)$.[5,6] Cox showed that, under more general probability vectors, CNE on the unit interval exist if and only if $c(\mathbf{p}, m) \leq 1/2$, where $c(\mathbf{p}, m) = \frac{p_1 - \bar{p}}{p_1 - p_m}$ and $\bar{p} = \frac{1}{m}(p_1 + \ldots + p_m)$ is the average score. The parameter $c(\mathbf{p}, m)$, also called the c-value, is always between 0 and 1 and is a measure of the dominating incentives (Myerson, 1999). Cox also considered a two-dimensional market, making several important observations that we use here, including an indirect characterisation of CNE.

Here we combine the two directions of generalisation described above, showing how CNE depend on both the market shape and consumer shopping be-

[4] We allow $p_1 + \ldots + p_m < 1$. The results are invariant to affine transformations of $\mathbf{p}$. Therefore for readability we often scale vectors by a convenient constant.

[5] In the voting context, such vectors are known as *scoring rules*, and include plurality rule, antiplurality and Borda rule, respectively. This is why we will sometimes refer to the probability vector $\mathbf{p}$ as a *rule*.

[6] This is not the first paper to consider probabilistic voting in general—see, for example, Anderson et al. (1994) or De Palma et al. (1990). The approach there, however, is otherwise very different to our setting.

haviour. We begin with the benchmark case of a two-dimensional convex square market—any location in the square can be chosen by a firm to set up shop without restrictions. We provide a new geometric proof of Cox's (1987) indirect characterisation that shows CNE may exist for a worst-punishing or intermediate probability vector in a sufficiently central region of the market bounded by hyperbolas.

Noting that a city's layout is decidedly non-convex and has more in common with a network of nodes and edges, we then proceed to study the properties of CNE on general networks. On a network, new considerations arise. Firstly, instead of Euclidean distances, we must analyse shortest paths between points on the network. Secondly, while previous papers study settings where positive masses of indifferent consumers do not arise, or disregard the possibility as unrealistic, on a network they occur naturally (see, e.g., Figure 1) and add another layer of complexity to the analysis. We study in detail the case of a square grid, which illustrates explicitly these phenomena. In general, more central areas are able to be supported as CNE for a broader range of probability vectors $\mathbf{p}$, as in the case of an unrestricted two-dimensional square. However, at a more local level, the equilibrium properties of edges (interior locations on streets) exhibit quite different patterns and are more irregular compared to those on nodes (intersections of streets). On top of our characterisation of CNE for grids, the results suggest that a greater focus on atomic distributions, even on unidimensional spaces, may be an important and fruitful direction for future research.

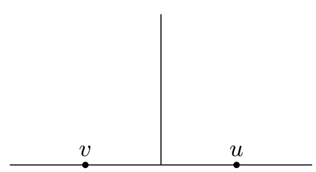

Fig. 1: On this network, all consumers on the vertical edge—a positive mass—are indifferent between points u and v.

2 CNE for Measurable Metric Location Spaces

2.1 General Results

In this section we will give a general result on the existence of CNE and consider some examples of particular markets. Later we investigate in detail two specific markets—a convex square space and a grid.

The location space, denoted I, is where both consumers and firms locate. It is assumed that the location of a consumer is fixed while the firms compete by choosing locations (and that the relocation costs are negligible). We assume that the location space I is a measurable metric space. The metric determines preferences: if there are m competing firms located on I in positions $u_1, \dots, u_m$ and $v \in I$ is the location of a consumer, then the consumer has then a linear preference order $\succeq_v$ on the set of firms determined by distance along the shortest path (closer firms are preferred to more distant ones). If two or more firms are equidistant from the consumer, she breaks the tie at random.

There is also a measure μ defined on I which gives the density of consumers on subsets of I. We assume I has finite measure, i.e., $\mu(I) < \infty$. Let us introduce the following notation:

$$K_{u>v} = \mu(\{\text{consumers with strict preference for } u \text{ over } v\}),$$

where $u, v \in I$. $K_{u\sim v}$, analogously, is the set of indifferent consumers. The measure μ is assumed to be such that these sets are always measurable. Given a vector of probabilities $\mathbf{p}$, each firm then gets a score equal to the expected measure of customers who will patronise it. Firms choose their locations simultaneously so as to maximise this expected measure of customers. Our solution concept is the Nash equilibrium, which is said to be convergent if all firms adopt the same location.

For any $u \in I$, we define

$$K(u) = \sup_{v \in I} \frac{K_{v>u}}{K_{u>v}}.$$

This is the maximal achievable ratio of consumers preferring some other point v to u over consumers that prefer u to v.

The main result of this section is Theorem 1. Subsequently when we focus on the convex square market and networks we will focus on uniform measures for tractability, but Theorem 1 holds in general.[7] For the square convex market case, the result was proven by Cox (1987). Theorem 1 is more general, however, because it applies on other spaces where the measure of consumers indifferent between two points may be non-zero. In fact, this generality is critical for our study because non-zero measures of indifferent consumers are a regular occurrence on networks.

Theorem 1. *Given a location space I with m competing firms and probability vector $\mathbf{p}$, a point $u \in I$ is a CNE if and only if*

$$K(u) \le \frac{1}{c(\mathbf{p}, m)} - 1. \tag{1}$$

Proof. If all firms are located at u, each of them will get $\overline{p}\mu(I)$ consumers. If a single firm deviates from u to location v, then all customer will be split into

[7] In our contexts, reasonable non-uniform distributions of consumers would thus retain the flavour of the results, but the specific formulas we derive would no longer apply. Note that uniform distributions are not necessarily unrealistic—they are consistent with firms that believe or assume the distribution is uniform as discussed by Aragonès and Xefteris (2012) and Cahan and Slinko (2017).

three sets: those who are closer to v than to u, those equidistant from v and u and those who are closer to u than to v. The expected number of customers shopping at the deviating firm would be $p_1 K_{v>u} + \overline{p} K_{u\sim v} + p_m K_{u>v}$.

Thus, for any $v \in I$, for the deviation to v not to be profitable, we must have

$$p_1 K_{v>u} + \overline{p} K_{u\sim v} + p_m K_{u>v} \le \overline{p}\mu(I) = \overline{p}(K_{v>u} + K_{u\sim v} + K_{u>v}).$$

This rearranges to

$$\frac{K_{v>u}}{K_{u>v}} \le \frac{\overline{p} - p_m}{p_1 - \overline{p}} = \frac{p_1 - p_m + \overline{p} - p_1}{p_1 - \overline{p}} = \frac{1}{c(\mathbf{p}, m)} - 1.$$

For u to be a CNE, even the best deviation should not be profitable, hence we take the supremum of the left-hand side in (1). □

Note that the value $K(u)$ is not always defined, e.g., when u is a leaf on a network. Then we set $K(u) = \infty$. Hence Theorem 1 rules out a CNE at a leaf.

If the measure of indifferent consumers is always zero, which will be the case on a line or a two-dimensional square (but not on grids), it will be more convenient to use the value $W(u) := \sup_{v\in G} \frac{K_{v>u}}{K} = \frac{K(u)}{1+K(u)}$, where K is the total measure of consumers. Then condition (1) can be rewritten as

$$W(u) \le 1 - c(\mathbf{p}, m). \tag{2}$$

Example 1 (Linear, circular and star markets). For a linear market $I = [0, 1]$ with a uniform distribution of consumers, CNE exist if and only if $c(\mathbf{p}, m) \le 1/2$. Any location $x \in I$ such that $c(\mathbf{p}, m) \le x \le 1 - c(\mathbf{p}, m)$ can support a CNE (Cox, 1987).

For a circular market I with a uniform distribution of consumers, a CNE exists if and only if $c(\mathbf{p}, m) \le 1/2$. In such a case every location can support a CNE. Indeed, deviating to a point $v \neq u$ a deviating firm gets half of the customers, so $K(u) = 1$.

Consider a star graph S_k with $k \ge 2$ edges and $k + 1$ nodes where each edge has measure 1 of consumers. The central node u is a CNE if and only if $W(u) = 1/k \le 1 - c(\mathbf{p}, m)$. This is because the best deviation for any firm would be to deviate marginally towards one of the leaves. In particular, in the classical case with $\mathbf{p} = (1, 0, \dots, 0)$, it is a CNE if and only if $m \le k$, i.e., the number of firms does not exceed the number of edges. For any worst-punishing or intermediate $\mathbf{p}$, it is always a CNE. If u is on an edge at distance ϵ away from the central node, then the best deviation will be to deviate to the point v marginally towards the central node, in which case $K_{v\succ u} = k - 1 + \epsilon$ and $W(u) = 1 - \frac{1-\epsilon}{k}$. Then (2) tells us that for u to be a CNE we must have $c(\mathbf{p}, m) \le \frac{1-\epsilon}{k}$. In particular, it will never be a CNE for $\mathbf{p} = (1, 0, \dots, 0)$, but for $\mathbf{p} = (1, \dots, 1, 0)$ it will always be when the number of firms is large enough: $k \le m(1 - \epsilon)$.

2.2 Two-dimensional square market

We start by considering a two-dimensional square without any restrictions on where firms can locate as perhaps the most basic extension of the linear model.

This serves as a benchmark for our subsequent study of what happens when restrictions are placed on where firms and consumers are located, i.e., when we look at a grid. This case has previously been studied by Cox (1987), who showed that CNE exist if and only if $c(\mathbf{p}, m) \leq 1/2$. However, this indirect result does not explicitly describe the set of points that can support CNE for a given probability vector—here, we provide a direct, geometric result.

There cannot be a positive measure of consumers indifferent between two points, so for a point $u = (x, y)$, we only need to calculate the simpler $W(u) = \sup_{v \in I} K_{v>u}/K = \sup_{v \in I} K_{v>u}$ (normalising the total measure of consumers to $K = 1$). We derive an explicit formula for $W(u)$. This will allow us to find a region R for which points inside of R are CNE for a given $\mathbf{p}$. We assume without loss of generality that $u \in I_1 = \{(x, y) \in I : 0 \leq x \leq 1/2, 0 \leq y \leq 1/2\}$.

It is clear that the best deviation by a single firm from u is to a point infinitesimally close to u in some direction. Then $W(u)$ will be the area of one of the regions which appear when a line through u divides the square into two parts.

Lemma 1. *Let λ be a line through u in the interior of I_1 dividing the square into regions α and β so that $\mu(\beta) = W(u)$ and $\mu(\alpha) = 1 - W(u)$. Then α is a triangle.*

Lemma 2. *For any $u = (x, y) \in I_1$ we have $W(u) = 1 - 2xy$.*

This means that we have our desired explicit formula for $W(u)$, namely $W(u) = 1 - 2xy$ for $u \in I_1$. And since I is symmetric we have similar formulas for u in the other quadrants as well. This allows us to fully characterise CNE geometrically in the square market.

Theorem 2. *For a given probability vector p with $c(\mathbf{p}, m) \leq \frac{1}{2}$, the profile $u = (u, \ldots, u)$ is a CNE if and only if $u \in R = \bigcup\{R_1, R_2, R_3, R_4\}$, where:*

$$R_1 = \{(x, y) \in I_1 : 2xy \geq c(\mathbf{p}, m)\};$$
$$R_2 = \{(x, y) \in I_2 : 2(1 - x)y \geq c(\mathbf{p}, m)\};$$
$$R_3 = \{(x, y) \in I_3 : 2(1 - x)(1 - y) \geq c(\mathbf{p}, m)\}; \text{ and}$$
$$R_4 = \{(x, y) \in I_4 : 2x(1 - y) \geq c(\mathbf{p}, m)\}.$$

Proof. If $u = (\frac{1}{2}, \frac{1}{2})$ then by Lemma 2 u is a NCNE. If $u \in I_1$, then by Theorem 1 and Lemma 2 we must have that u is a CNE if and only if $2xy \geq c(\mathbf{p}, m)$, or equivalently $u \in R_1$. Since R must be symmetric about the axis of symmetry of I, we must have that if $u \in I_2 = \{z \in I : \frac{1}{2} \leq z_1 \leq 1, 0 \leq z_2 \leq \frac{1}{2}\}$ then u is a CNE if and only if $u \in R_2$ too. Similarly for all i if $u \in I_i$ then u is a CNE if and only if $u \in R_i$. □

Therefore we have found a region in I bounded by four hyperbolas for which all points inside are CNE and all points outside are not CNE. The region in which CNE exist is illustrated in Figure 2. Note that, as $c(\mathbf{p}, m)$ decreases, the region R expands and CNE can be supported at points further than the center of the

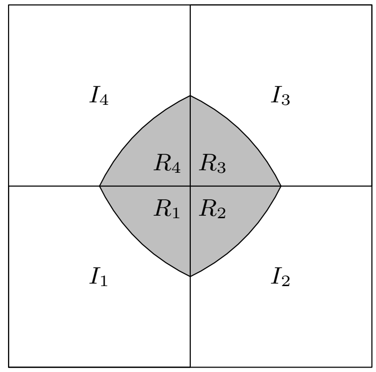

Fig. 2: Illustrating the area R of CNE for $c(\mathbf{p}, m) = 1/4$.

square. Moreover, except for a point on the boundary, any point can be a CNE for low enough $c(\mathbf{p}, m)$. This pattern will be roughly reflected when we restrict the space to a grid, but not exactly—the non-convexity of the grid, specifically the fact that you can only reach a firm through a limited number of paths, will give rise to new equilibrium phenomena.

3 Networks

3.1 General results for networks

Let $G = (V, E)$ be a graph consisting of a set of nodes (equivalently, "vertices") V and a set of edges E. We assume a uniform distribution of consumers over the network, normalised so that one edge has measure 1. We will call the set points on the edges (including nodes) of this network also G, this will lead to no confusion. According to graph theory, the distance between two nodes in a graph is the number of edges in a shortest path connecting them. We will consider the distance between any two points on a network, nodes or points on edges, in an analogous way.

As a further consequence of Theorem 1, Proposition 1, shows that for any point $u \in G$, except for a leaf, there is always a threshold value of $c(\mathbf{p}, m)$ below which no CNE exist. Thus any point can be a CNE given the right probability vector $\mathbf{p}$. Moreover, for certain $\mathbf{p}$ no CNE can be supported at any point on the network.

Proposition 1. *Let m be fixed. For any point $u \in G$, except for a leaf, there exists a critical value $\bar{c} = \bar{c}(u)$ such that:*

(i) The set of all rules $\mathbf{p}$ with $c(\mathbf{p}, m) > \bar{c}$ do not admit CNE at u.
(ii) The set of all rules $\mathbf{p}$ wilth $c(\mathbf{p}, m) \leq \bar{c}$ do admit CNE at u.

Furthermore, there is a critical value $\overline{c}(G)$ such that: if $c(\mathbf{p}, m) > \overline{c}(G)$ there are no CNE at any point on the network; and if $c(\mathbf{p}, m) \leq \overline{c}(G)$, there is at least one CNE on the network.

Proof. Inequality (1) can be rewritten as $c(\mathbf{p}, m) \leq \frac{1}{1+K(u)}$. Then $\overline{c}(G) = 1/(1+\inf_{u\in G} K(u))$ would be the threshold value for G. □

An alternative way to look at Proposition 1 is taking m and $\mathbf{p}$ as fixed and varying the graph. Then, CNE will always exist for nodes u that are "central" enough in the sense of a low value of $K(u)$. E.g., as seen in Example 1, for a star graph the central node has $K(u) = {}^1\!/_{k-1}$, which tends to zero as k gets large.

Example 2. Certain probability vectors are defined recursively for any m, such as $\mathbf{p} = (1, 0, \ldots, 0)$, $\mathbf{p} = (1, \ldots, 1, 0)$ or $\mathbf{p} = (m-1, m-2, \ldots, 1, 0)$. Such vectors can be considered sequences $(\mathbf{p}^{(m)})_{m\in\mathbb{N}}$ where $\mathbf{p}^{(m)}$ corresponds to the case of m firms. A probability vector is *asymptotically best rewarding* if, as $m \to \infty$, $c(\mathbf{p}^{(m)}, m) \to 1$ and *asymptotically worst punishing* if $c(\mathbf{p}^{(m)}, m) \to 0$. For the three vectors above $c(\mathbf{p}^{(m)}, m)$ tends to 1, 0 and ${}^1\!/_2$, respectively. When m gets large enough, an asymptotically best rewarding vector cannot sustain CNE, while an asymptotically worst punishing rule will allow CNE at any point except for a leaf when m is large enough.

Fournier and Scarsini (2019) find that, if the number of firms m is sufficiently large, there will be a *non-convergent* Nash equilibrium (NCNE) under $\mathbf{p} = (1, 0, \ldots, 0, 0)$ for every network G. Applying Proposition 1 provides a counterpart to their result. It implies that all equilibria must then be NCNE because no CNE are possible once m is large.

3.2 Grids

We now turn our attention to a specific kind of network: the grid. This is perhaps the most natural network to represent a stylised district of the city. Unlike the convex square, this market imposes the more realistic assumption that activity in the city occurs along streets – firms cannot locate anywhere they please, and consumers do not travel as the crow flies.

Let the graph G be a square grid with sides that are M edges long. Thus, it has $(M+1)^2$ nodes and $2M(M+1)$ edges in total. To keep things simple, we will focus on the case where M is even—insights are similar in the case of M odd. Let $u \in G$ be a location in the northeast quadrant of the grid (by symmetry, we can restrict attention to this area of the grid). We write $u = (x, y)$, where $x \geq 0$ is the horizontal distance from u to the central node and $y \geq 0$ is the vertical distance. Thus, the central node is at coordinates $(0, 0)$ and the northeast corner is at $({}^M\!/_2, {}^M\!/_2)$. Since the network includes edges as well as nodes, either x or y can be a non-integer, but not both.

Consider nodes $u = (x, y)$ and $u' = (x-1, y-1)$ where $y, x \geq 1$. Note that consumers in the northwest and southeast corners of the grid are indifferent between u and u' (see Figure 3). Let us introduce the notation:

$$NW(u) = \{(x', y') \in K_{u \sim u'} : x' \leq x \text{ and } y' \geq y\};$$
$$SE(u) = \{(x', y') \in K_{u \sim u'} : x \leq x' \text{ and } y \geq y'\}.$$

Though functions of u, we usually just write NW and SE when no confusion can emerge. We can write down an explicit formula for the number of consumers in each of these sets:

$$|NW| = (M/2 + x - 1)(M/2 - y + 1) + (M/2 + x)(M/2 - y),$$
$$|SE| = (M/2 + y)(M/2 - x) + (M/2 + y - 1)(M/2 - x + 1).$$

In a similar way, note that the consumers in the northeast portion of the grid described in Figure 3 strictly prefer u over u', while those in the southwest portion prefer u' over u. We use the notation $|SW| = |K_{u'>u}|$ and $|NE| = |K_{u>u'}|$. We have the following formulas for these areas:

$$|SW| = 2(M/2 + x)(M/2 + y),$$
$$|NE| = 2(M/2 - x + 1)(M/2 - y + 1).$$

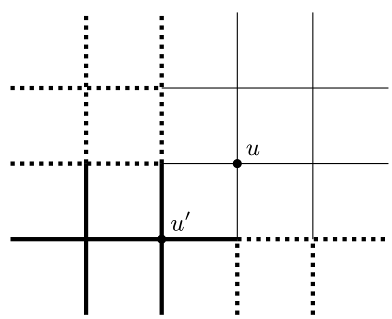

Fig. 3: Thick black lines indicate consumers preferring u' to u; thin black lines: consumers preferring u to u'; dotted lines: consumers indifferent between u' and u.

To prove that a certain point is a CNE, we need to show that *no* possible deviation by a single firm is advantageous. The following result determines one or several potential "best deviations" that dominate all other possible deviations. The partition of the grid into northwest, northeast, southeast and southwest corners (relative to u) defined above is the key determinant of what is the "best deviation" from the node u, that is, the deviation u' that maximises the ratio $K_{u'>u}/K_{u>u'}$. Once we know these best deviations we can calculate $K(u)$ and thus determine whether a CNE at point u exists.

Theorem 3. *Let $u = (x, y)$ be a node with $x, y \geq 0$ and m firms co-located at u. Then the deviation u' that maximises the ratio $K_{u'>u}/K_{u>u'}$ is among the following:*

(i) If $x > 0$ and $y > 0$, the point $(x-1, y-1)$ or just to the east or north of it.
(ii) If $y = 0$ and $x > 0$, the point $(x-1, y)$. By symmetry, the point $(x, y-1)$ if $x = 0$ and $y > 0$.
(iii) If $u = (0, 0)$, either $(1, 0)$ or $(0, 1)$.

In case (i),

$$K(u) = \max\left\{\frac{|NW| + |SW|}{|NE| + |SE|}, \frac{|SW|}{|NE|}, \frac{|SW| + |SE|}{|NW| + |NE|}\right\}. \tag{3}$$

In case (ii),

$$K(u) = \frac{M^2 + (2M+1)\max(x, y) - 1/2}{M^2 + 2M - (2M+1)\max(x, y) + 1/2}. \tag{4}$$

In case (iii),

$$K(u) = \frac{M^2 - 1/2}{M^2 + 2M + 1/2}. \tag{5}$$

Thus, u can be supported as a CNE if and only if $K(u) \leq \frac{1}{c(\mathbf{p},m)} - 1$ and the threshold value of $c(\mathbf{p}, m)$ below which u can be supported as a CNE is $\frac{1}{K(u)+1}$.

Example 3. Let us consider the example of $M = 6$, displayed in Figure A.5. We calculate the threshold c-value (the c-value above which no CNE exist) for each node in the top right quadrant in Figure A.5 (the remaining nodes follow by symmetry). This does not say anything about whether CNE exist on interior points of an edge (non-nodes).

Theorem 3 thus characterises CNE at nodes on the grid. Next, we turn our attention to points *interior* to an edge. Let $u = (x - \epsilon, y)$, where $x, y \geq 1$ and $0 < \epsilon < 1$, and let $u' = (x - 1, y - 1 + \epsilon)$. Given these two points, again the grid can be partitioned into four regions: two regions of indifferent consumers in the northwest and southeast and, in the northeast and southwest, consumers preferring u and u', respectively (see Figure A.6(a)). Let NW', NE', SE' and SW' denote these regions and let NW, NE, SE and SW be the areas defined above during the comparison of (x, y) with $(x - 1, y - 1)$. Then we have the following formulas:

$$|NW'| = |NW| + \epsilon(M + x - y - 1); \qquad |SE'| = |SE|;$$
$$|SW'| = |SW| - \epsilon(M/2 + x - 1); \qquad |NE'| = |NE| - \epsilon(M/2 - y).$$

Note that the case where $u = (x, y - \epsilon)$ follows by symmetry.

Another important point to consider will be $u' = (x - 1 - \epsilon, y - 1)$, when $x > 1$ and $y \geq 1$. In a similar fashion, we again partition the grid into areas of indifferent consumers in the northwest and southeast, consumers who prefer u in the northeast and consumers who prefer u' in the southwest (see Figure A.6(b)). The number of consumers in these areas, denoted NW'', SW'', SE'' and NE'', are given by the following formulas:

$$|NW''| = |NW| - M + 2y - 1; \qquad |SE''| = |SE|;$$
$$|SW''| = |SW| - \epsilon; \qquad |NE''| = |NE| + M - 2y + 1 + \epsilon.$$

Theorem 4. *Let $u = (x - \epsilon, y)$, where $x > 0$ and $0 < \epsilon < 1$, be occupied by m firms. The deviation u' of a single firm that maximises $K_{u'>u}/K_{u>u'}$ is among the following:*

(i) If $y > 0$ and $x > 1$, the point $(x-1, y-1+\epsilon)$, just to the north it, or the point just to the east of $(x-1-\epsilon, y-1)$.
(ii) If $y > 0$ and $x = 1$, the point $(x-1, y-1+\epsilon)$, just north of it, or the point $(x-1, y-1)$.
(iii) If $y = 0$, the point $(x-1, 0)$.
(iv) Cases that are symmetric to (i)–(iii).

Therefore, in case (i),

$$K(u) = \max\left\{\frac{|NW'| + |SW'|}{|NE'| + |SE'|}, \frac{|SW'|}{|NE'|}, \frac{|SW''| + |SE''|}{|NW''| + |NE''|}\right\}. \tag{6}$$

In case (ii),

$$K(u) = \max\left\{\frac{|NW'| + |SW'|}{|NE'| + |SE'|}, \frac{|SW'|}{|NE'|}, \frac{K_{\overline{u}>u}}{K_{u>\overline{u}}}\right\}, \tag{7}$$

where $\overline{u} = (x-1, y-1)$ and

$$\frac{K_{\overline{u}>u}}{K_{u>\overline{u}}} = \frac{(M/2 + y)(2M+1) - 1/2(M+\epsilon)}{(M/2 - y)(2M+1) + 3M/2 + \epsilon/2}. \tag{8}$$

In case (iii),

$$K(u) = \frac{(M/2 + x)(2M+1) - M/2 - 1/2 + \epsilon(M-1)/2}{(M/2 - x)(2M+1) + 3M/2 + 1/2 - \epsilon(M-1)/2}. \tag{9}$$

Thus, u is a CNE if and only if $K(u) \leq \frac{1}{c(\mathbf{p},m)} - 1$ and the threshold value of $c(\mathbf{p}, m)$ below which u can be supported as a CNE is $\frac{1}{K(u)+1}$.

Theorem 4 characterises CNE at points on the interior of an edge. The formulas for $K(u)$ resemble those obtained in Theorem 3 except that there is an additional ϵ term on the numerator and denominator. Before we look at some of the implications of this, we state some properties of $K(u)$ in Corollary 1.

Corollary 1 says that, at any point u except for the central node $(0, 0)$, increasing M increases the threshold $\overline{c}(u)$, it becomes easier to support, as it is increasingly central in the network. Increasing x or y, for fixed M, on the other hand, decreases $\overline{c}(u)$. This similarly makes sense—as we move outward on a network, the point becomes less central and thus can be supported for a smaller range of probability vectors $\mathbf{p}$.

Corollary 1. *Let u be a point on the grid. Then $K(u)$ is:*

(i) Increasing in M, holding $u = (0, 0)$ fixed;
(ii) Decreasing in M, holding $u \neq (0, 0)$ fixed;

(iii) Increasing in x or y, holding M fixed.

At all points u, $K(u)$ converges towards 1 as M tends to infinity.

Corollary 1 shows how $K(u)$ depends on M, x and y. Next, consider how it depends on ϵ. Here, the dynamics are surprisingly complex, and the patterns are not unambiguous. In many cases, as $c(\mathbf{p}, m)$ decreases, the set of points along an edge supporting CNE "grows inwards" towards the center. That is, within an edge, points *further away* from the center or a central axis can be *easier* to support than points closer to the center.

Example 4. We return to the case of $M = 6$ and consider edges. Using Theorem 4 we can calculate how the threshold value $\bar{c}(u)$ varies along edges. The arrows in Figure A.7 point in the direction of decreasing $\bar{c}(u)$. In other words, from points that are easier to support (have a higher threshold) to points that are more difficult to support. In general, within a given edge, as we move "towards the center/towards an axis" of the grid, points become harder to support. However, not for the outermost edges (where $y = M/2$ or $x = M/2$). Figures A.8–A.11 plots the threshold value of $c(\mathbf{p}, m)$ as x and ϵ change for fixed values of y. Another striking feature apparent in these figures is the discontinuities in the threshold value of $c(\mathbf{p}, m)$ at nodes when ϵ approaches 1. For several illustrative values of $c(\mathbf{p}, m)$, the set of CNE, including edges, is displayed in Figure A.12. This demonstrates how the set of CNE changes with $c(\mathbf{p}, m)$ and how it may "grow inwards" along edges towards the center of the grid as described above.

4 Conclusion

In a real city firms are typically not restricted to just a linear "Main Street" when deciding where to set up shop, but rather the space of possible locations is two-dimensional. Even then, they are two-dimensional in a very restricted sense—firms locate along streets, the same streets that consumers must travel in order to shop. At the same time, most previous papers assume that consumers always purchase from the nearest firm when, more realistically, consumers are only *more likely* to purchase at nearer firm.

We study the existence and properties of CNE in a spatial setting that incorporates both of these more realistic elements. The results, aside from providing detailed new characterisations of CNE, point to interesting new phenomena that future research should look to explore further in a broader range of contexts. For example, in our study of networks, positive masses of indifferent consumers arise and play an important role. This suggests that allowing for the possibility of point masses of consumers, which are often assumed away in the previous literature, may be important more generally. In the workhorse one-dimensional Main Street model, for example, it may make a great deal of sense to allow atomic distributions of customers if there are certain access points through which consumers arrive to the market, which may be fiercely contested by the firms.

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Appendix

Proof (Lemma 1). Since $\mu(\alpha) \leq \mu(\beta)$, then α can be either triangle or a trapezium. Let λ be a line through u such that it forms a trapezium with the parallel bases being parts of each of the top and bottom sides of the square and one of the lateral sides being a side of the square. Now consider the line through u and the top left corner $(0, 1)$. Clearly the two light grey triangles (see Figure A.1) formed are similar. The "height" of the upper triangle (i.e., the distance from u to the upper side) is equal to $1 - y$, but, as $y \leq 1/2$, we know $1 - y \geq y$. Since y is the height of the lower triangle, the upper one is larger. This means that the triangle ABC must have smaller area than the trapezium $ABDE$, which shows that $\lambda \neq ED$. Therefore, α cannot be a trapezium. The arguments for horizontal trapeziums (see Figure A.1) are symmetric. Therefore we must have that α is a triangle. □

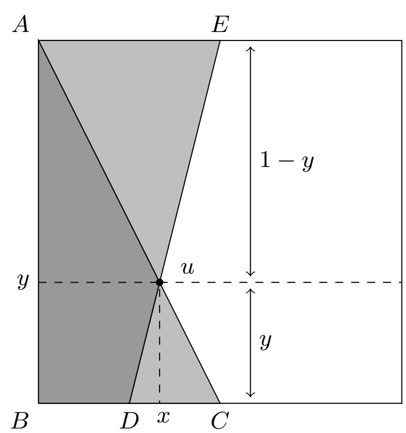


Fig. A.1: Dividing the square into triangles and trapeziums.

Proof (Lemma 2). If $y = 1/2$ the result is clear. Suppose now that u is in the interior of I_1. We let λ be the line through u which bounds α, the region as defined in the previous lemma. From Lemma 1 we also know that α must be a triangle, so we know that λ crosses the x- and y-axes. Let b be the intersection of λ with the y-axis and let a be the intersection with the x-axis, so that $\mu(\alpha) = \frac{1}{2}ab$. The gradient of λ is $m = \frac{y-b}{x}$, so that $\lambda = \{(x, y) \in I : y = mx + b\}$. We then have $0 = \frac{y-b}{x}a + b$, which is equivalent to $a = \frac{bx}{b-y}$. Therefore

$$\mu(\alpha) = \frac{1}{2}ab = \frac{b^2 x}{2(b-y)}, \text{ hence } \quad \frac{\partial \mu(\alpha)}{\partial b} = \frac{x}{2}\left(\frac{2b}{b-y} - \frac{b^2}{(b-y)^2}\right).$$

So then $\frac{\partial \mu(\alpha)}{\partial b} = 0$ if and only if $\frac{2b}{b-y} = \frac{b^2}{(b-y)^2}$, or equivalently $b = 2y$, and hence $a = 2x$. Therefore $\mu(\alpha) = \frac{1}{2}ab = 2xy$, and hence $W(u) = 1 - 2xy$, as required. □

Proof (Theorem 3). To determine whether u is a CNE we want to calculate $K(u)$. This requires finding the point u' at which the ratio $K_{u'>u}/K_{u>u'}$ is maximised. First, suppose $u = (x, y)$ is a node with positive integers x, y. We claim that the horizontal move (keeping y constant) that maximises $K_{u'>u}/K_{u>u'}$ is to the node $(x-1, y)$, rather than to a point in the interior of the edge, i.e., to a point $(x-\epsilon, y)$, where $0 < \epsilon < 1$. In Figure A.2(a), the customers on bold segments strictly prefer u' over u, while those on thinly shaded edges strictly prefer u to u'. The set of indifferent consumers has measure zero in such a configuration. Consider a point slightly to the left of u', i.e., $u'' = (x-\epsilon-\eta, y)$, where $\eta > 0$ is very small. We compare the score of i upon deviation to u' with the score upon deviation to u''. We can see that u'' looks more attractive than u' because the firm gains $\frac{\eta}{2}$ consumers from M edges and loses $\frac{\eta}{2}$ from only one edge (the one on which it is moving). Thus, the ratio $K_{u'>u}/K_{u>u'}$ is increasing as u' moves to the left. By the same logic, the firm would want to continue left until reaching node $(x-1, y)$ as there is no discontinuity of the score when reaching this node.[8]

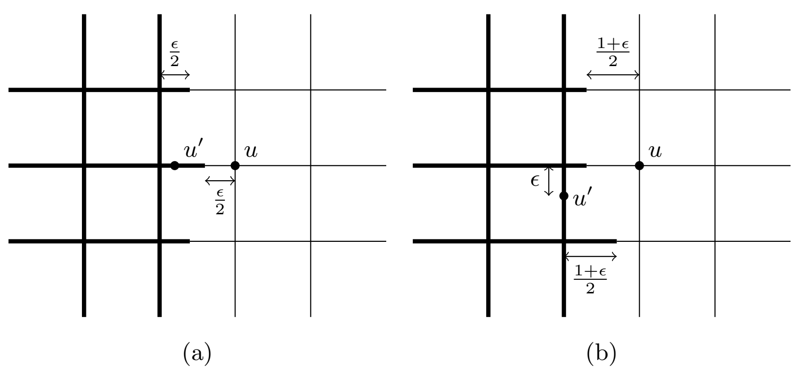


Fig. A.2: Possible deviations from a point u in case (i). Thick black lines indicate consumers preferring u' to u; thin black lines: consumers preferring u to u'.

However, the best horizontal may not be the best move overall. Consider a move to a point $u' = (x-1, y-\epsilon)$, where $0 < \epsilon < 1$. Consider also a point $u'' = (x-1, y-\epsilon-\eta)$ where $\eta > 0$ is small (see Figure A.2(b)). Moving from u' to u'', i is gaining from each edge $(x-1, z)$ where $z < y$ but losing at the same rate from each edge $(x-1, z)$, where $z \geq y$. Because $y > 0$, there are more

[8] Clearly, a move even further along would decrease $K_{u'>u}/K_{u>u'}$—once we pass the node $(x-1, y)$, moving further to the left implies a loss along all the edges between $(x-1, z)$ and (x, z) for all z.

edges from which i is gaining, so u'' looks better than u'. Thus, $K_{u'>u}/K_{u>u'}$ increases as i moves towards the next node $(x-1, y-1)$.

By a similar argument, firm i could also have, instead of initially moving horizontally from u, moved down vertically from u to the node $(x, y-1)$ and then across towards $(x-1, y-1)$ from the right.

However, whether i should actually arrive at the node $(x-1, y-1)$ itself is not clear, since there is a potential discontinuity in the score. The reason is that, at $(x-1, y-1)$, the areas NW and SE will be indifferent between this node and u (the dotted edges in Figure 3), and these are sets of positive measure. We therefore need to compare the ratio $K_{u'>u}/K_{u>u'}$ for three possible deviations: $u' = (x-1, y-1)$, the limiting point just north of $(x-1, y-1)$ and the limiting point just east of $(x-1, y-1)$.[9]

Next, consider the cases where we are on one of the central axes, i.e., $u = (x, 0)$ or $u = (0, y)$. If $x > 0$ and $y = 0$, a deviation to u' between u and $(x-1, 0)$ results in a higher value of $K_{u'>u}/K_{u>u'}$ if u' is further from u. In fact, the best move would be to the point $(x-1, 0)$. Finally, if $u = (0, 0)$, similar reasoning leads to the conclusion that the best move is to $u' = (1, 0)$ or $u' = (0, 1)$. □

Proof (Theorem 4). First, let us consider case (i). Consider the point $u = (x-\epsilon, y)$, where $x > 1$ and $y > 0$ are integers and $0 < \epsilon < 1$. Consider a firm deviating to $(x-\epsilon-\eta, y)$, where $\eta > 0$ is small. As in the case where u is the node (x, y), it would be better to deviate to the node $(x-1, y)$, because by moving to the left the firm is gaining from consumers along multiple edges and is losing from only one edge.

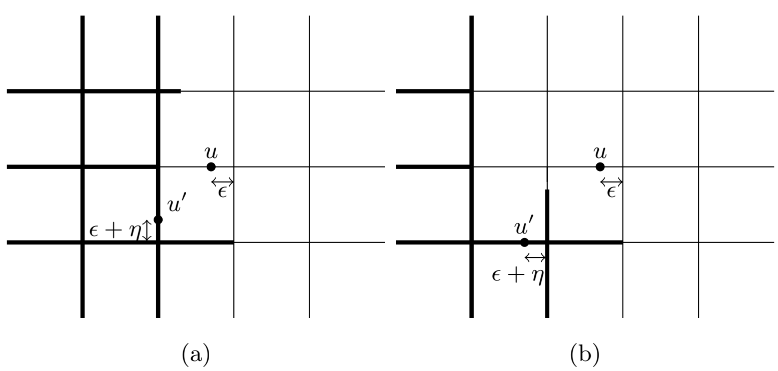


Fig. A.3: Comparing the move just north of $(x-1, y-1+\epsilon)$ to the move just west of $(x-1-\epsilon, y-1)$. Thick black lines indicate consumers preferring u' to u; thin black lines: consumers preferring u to u'.

[9] Note that if $y \geq x$, $|SE| \geq |NW|$, so the move to $(x-1+\epsilon, y-1)$ is better than the move to $(x-1, y-1+\epsilon)$ and we only need one comparison.

Upon reaching the node, the firm would then want to continue deviating towards $(x-1, y-1)$ because, again, there are more edges from which the firm is gaining consumers than from which it is losing. However, the firm will continue to gain only until it reaches the point $(x-1, y-1+\epsilon)$, at which point the northwest and southeast corners become indifferent (see Figure A.6(a)). Therefore, the move that maximises $K_{u'>u}/K_{u>u'}$ could be $(x-1, y-1+\epsilon)$, just to the north, or just south of it. However, the latter move is actually dominated since by continuing to move down the edge towards $(x-1, y-1)$ the number of edges from which the firm is gaining consumers exceeds those from which it is losing consumers. Even on reaching $(x-1, y-1)$, since $x > 1$, there is a further incentive to deviate west, to the point $(x-1-\epsilon, y-1)$. Here, there is another discontinuity in the score (see Figure A.6(b)) so the following points could also potentially be the best deviation: $(x-1-\epsilon, y-1)$, just to the east, or west of it.

It turns out, however, that of the latter three points, only the point just east of $(x-1-\epsilon, y-1)$ is not dominated by another deviation. To see this, first consider the deviation to a point inifinitesimally west of $(x-1-\epsilon, y-1)$. This is dominated by the deviation infinitesimally north of $(x-1, y-1+\epsilon)$. This can clearly be seen by comparing the two panels in Figure A.3.[10]

As for the deviation to precisely the point $(x-1-\epsilon, y-1)$, it is dominated by the move precisely to $(x-1, y-1+\epsilon)$. To see this, we show that $|SW'|/|NE'| > |SW''|/|NE''|$. That is,

$$\frac{|SW| - \epsilon(M/2 + x - 1)}{|NE| - \epsilon(M/2 - y)} > \frac{|SW| - \epsilon}{|NE| + M - 2y + 1 + \epsilon}.$$

This can be expressed as

$$|SW|(2+\epsilon)(M/2 - y + 1) + 2\epsilon|NE| > \epsilon^2(M/2 - y) + \epsilon(M/2 + x - 1)(M - 2y + 1 + \epsilon) + \epsilon|NE|(M/2 + x). \tag{10}$$

That inequality (10) is true follows from the following three facts: (a) $2|SW|(M/2 - y + 1) > \epsilon|NE|(M/2 + x)$; (b) $\epsilon|SW|(M/2 - y + 1) > \epsilon(M/2 + x - 1)(M - 2y + 1 + \epsilon)$; and, (c) $2\epsilon|NE| > \epsilon^2(M/2 - y)$. Inequality (a) can be written as

$$4(M/2 + x)(M/2 + y)(M/2 - y + 1) > 2\epsilon(M/2 - x + 1)(M/2 - y + 1)(M/2 + x)$$

which, cancelling terms, is clearly true. Inequality (b) can be rewritten as

$$2(M/2 + x)(M/2 + y)(M/2 - y + 1) > 2(M/2 + x - 1)(M/2 - y + (1+\epsilon)/2).$$

Again, this is clearly true. Finally, (c) states that $2\epsilon(M/2 - x + 1)(M/2 - y + 1) > \epsilon^2(M/2 - y)$, also true.

[10] Moreover, from the point just west of $(x-1+\epsilon, y-1)$, it would not be desirable to continue deviating along that edge because the number of edges from which the firm is losing consumers exceeds the number from which it is gaining consumers.

Hence, there are three possibilities for the best deviation from u: the move to $(x-1, y-1+\epsilon)$, just north of it, or just east of $(x-1-\epsilon, y-1)$. Thus, $K(u)$ will be given by equation (6) and we have proven case (i).

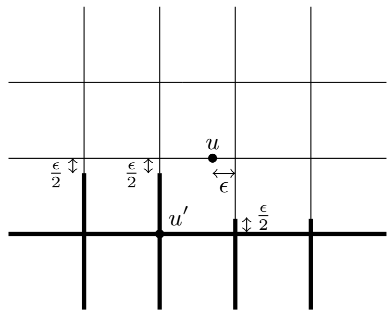


Fig. A.4: Possible deviation from a point u. Thick black lines indicate consumers preferring u' to u; thin black lines: consumers preferring u to u'.

For case (ii), where $x = 1$ and $y > 0$, the argument is similar to above. Two possibilities for the best deviation are the point $(x-1, y-1+\epsilon)$ and the point just north of it. The third possibility is the point $\overline{u} = (x-1, y-1)$—the firm would not continue west this time because it would gain and lose consumers from the same number of edges (see Figure A.4(e)). In case (ii), therefore, we have equations (7) and (8).

For case (iii), $u = (x-\epsilon, 0)$, where $x > 0$ and $0 < \epsilon < 1$. We show the move that maximises ${}^{K_{u'>u}}/_{K_{u>u'}}$ is to the point $(x-1, 0)$. Consider a deviation by i to a point further along the same edge $(x-\epsilon-\eta, 0)$ for very small $\eta > 0$. This is not the best deviation because i could continue further towards the node $(x-1, 0)$ and gain customers from all the parallel horizontal edges, while only losing customers from the edge on which it is located. It follows that the best deviation would be to $u' = (x-1, 0)$. In this case,

$$K(u) = \frac{|K_{u'>u}|}{|K_{u>u'}|} = \frac{({}^{M}/_{2} + x - 1)(M+1) + ({}^{M}/_{2} + x)M + M - (M-1)(1-\epsilon)/2}{({}^{M}/_{2} - x + 1)M + ({}^{M}/_{2} - x)(M+1) + 1 + (M-1)(1-\epsilon)/2}$$
$$= \frac{M^2 + (2M+1)x - {}^{1}/_{2} + {}^{\epsilon(M-1)}/_{2}}{M^2 + 2M - (2M+1)x + {}^{1}/_{2} - {}^{\epsilon(M-1)}/_{2}}. \square$$

Proof (Corollary 1). Case (i) follows by simply differentiating (5) with respect to M. For (ii), we first show that each of the arguments in the maximum function in equation (6) is decreasing in M. From this, the case where u is a node follow by setting $\epsilon = 0$ because (6) reduces to (3) (in fact, in the following proof (6) would be decreasing in M even allowing $x = 1$, which corresponds to nodes where $x = 1$).

Consider $|SW'|/|NE'|$. Differentiating with respect to M, the derivative is negative if and only if

$$(M - x - y - \epsilon/2 + 2)(2(M/2 + x)(M/2 + y) - \epsilon(M/2 + x - 1)) \\ - (M + x + y - \epsilon/2)(2(M/2 - x + 1)(M/2 - y + 1) - \epsilon(M/2 - y)) > 0.$$

This can be rewritten as

$$2(x + y - 1)(M(M + 2) - \epsilon(M - x + y + 3) + 2(x + y - 2xy) + \epsilon^2/2) > 0,$$

which is equivalent to

$$(x + y - 1)(M^2 + 2M - \epsilon M + \epsilon x - \epsilon y - 3\epsilon + 2x + 2y - 4xy + \epsilon^2/2) > 0.$$

By assumption $x + y - 1 > 0$ and the term in the second set of brackets can be written as

$$(M^2 - 4xy) + M(2 - \epsilon) + y(1 - \epsilon) + (2x + y - 3\epsilon) + \epsilon x + \epsilon^2/2.$$

That this is positive follows from the facts that $1 \leq x, y \leq M/2$ and $0 \leq \epsilon < 1$.

Next, consider $(|NW'|+|SW'|)/(|NE'|+|SE'|)$. This can be simplified to

$$\frac{(M/2 + x)(2M + 1) - M/2 + y - 1 + \epsilon(M/2 - y)}{(M/2 - x + 1)(2M + 1) - M/2 - y - \epsilon(M/2 - y)}.$$

Differentiating with respect to M, the derivative is negative if and only if

$$M^2(4x + \epsilon - 2) + (4M + 2)(x + y(1 - \epsilon) - 1) > 0,$$

which is certainly true.

Similarly, consider $(|SW''|+|SE''|)/(|NW''|+|NE''|)$. This can be simplified to

$$\frac{M^2 + 2My + x + y - \epsilon - 1}{M^2 - 2My + 2M - x - y + 1 + \epsilon}.$$

Differentiating with respect to M, the derivative is negative if and only if

$$(2y - 1)M^2 + 2M(x - \epsilon) + 2M(y - 1) + x + y - 1 - \epsilon > 0.$$

Given $y > 0$, $x > 1$ and $0 \leq \epsilon < 1$, this is certainly true.

Next consider the case where $u = (x - \epsilon, y)$ where $y > 0$ and $x = 1$. It is straightforward to show that, in a similar fashion to the above, (8) is decreasing in M. In particular, its derivative is negative if and only if

$$(4y - 2)M^2 + 4My + 2y(\epsilon + 1) - \epsilon > 0,$$

which is evidently true in this case.

Last, we consider the case where, without loss of generality, $y = 0$. The derivative of (9) will be negative if and only if

$$M^2(4x + \epsilon - 2) + (2M + 1)(2x - 1 - \epsilon) > 0,$$

which is always true.

To prove (iii), if $x > 0$, it is easy to see that $|SW'|$, $|NW'| + |SW'|$ and $|SW''| + |SE''|$ are all increasing in x, while $|NW'|$, $|NE'| + |SE'|$ and $|NW''| + |NE''|$ are decreasing in x. Thus, the ratios in (6) are all increasing. This is true even when $\epsilon = 0$ and when $x = 1$, so the case of nodes follows. The same holds true when y increases. We also need to check the case where $x = 1$, i.e., $u = (1 - \epsilon, y)$. Two of the arguments in the maximum in (7) are the same as in the previous case, hence increasing in x, and it can also be shown that (8) is less than $(|SW''(v)|+|SE''(v)|)/(|NW''(v)|+|NE''(v)|)$ for $v = (2 - \epsilon, y)$. Thus, (8) will be less than $K(v)$.

When $y = 0$, it is clear that (9) is increasing in x. Finally, we need to show that $K(u)$ is increasing when $x = 0$ and $y > 1$. By symmetry, consider instead $u = (x - \epsilon, 0)$ and $v = (x - \epsilon, 1)$. We show that $K(u) < K(v)$. Consider a deviation from v to just north of $v' = (x - 1, y - 1 + \epsilon)$. For this move,

$$\frac{K_{v'>v}}{K_{v>v'}} = \frac{|NW'(v)| + |SW'(v)|}{|NE'(v)| + |SE'(v)|} \leq K(v).$$

Using equation (9), after a bit of algebra it can be shown that $K(u) < K(v)$. □

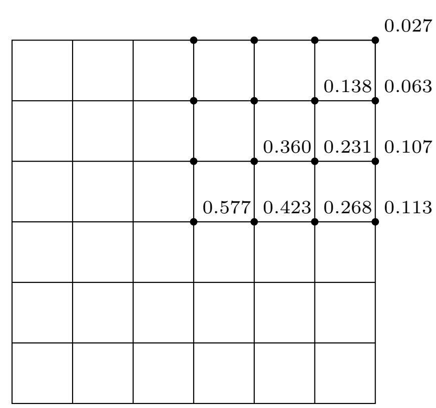


Fig. A.5: CNE on nodes in the case of $M = 6$. The numbers at the nodes of the grid indicate the threshold values $\bar{c}(u)$ corresponding to those nodes.

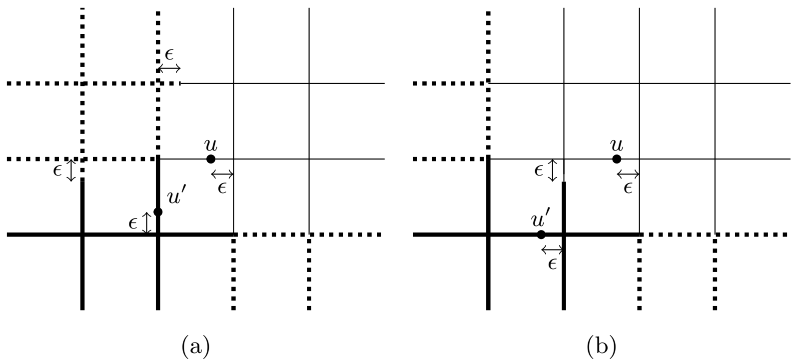


Fig. A.6: Possible deviations from a point u in the interior of an edge. Thick black lines indicate consumers preferring u' to u; thin black lines: consumers preferring u to u'; dotted lines: consumers indifferent between u' and u.

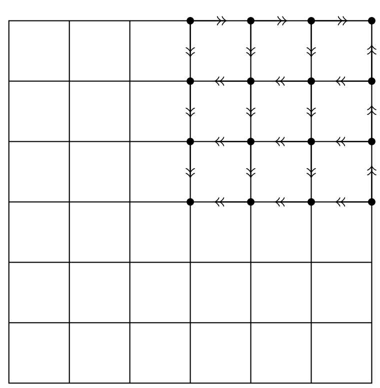


Fig. A.7: CNE on edges in the case of $M = 6$. For points in the interior of an edge (non-nodes), the arrows indicate the direction of decrease of the threshold values within the given edge.

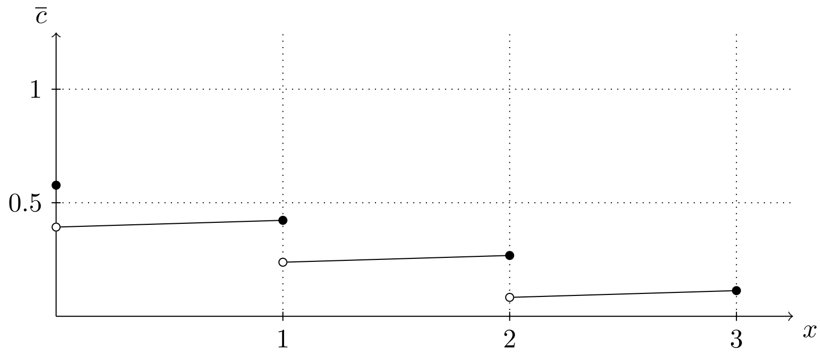


Fig. A.8: Threshold $\overline{c}$ along the segment from $(0, 0)$ to $(3, 0)$.

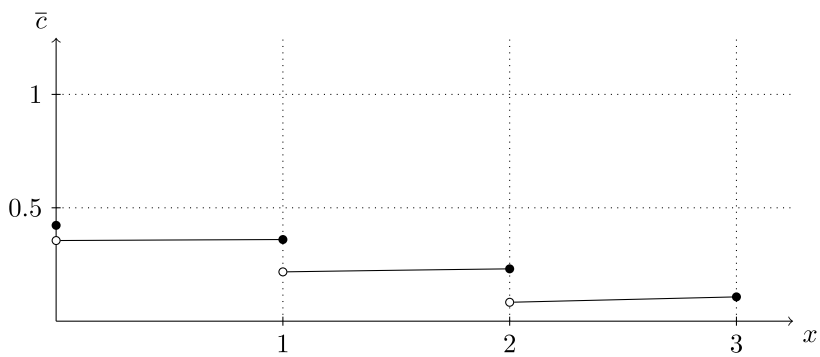


Fig. A.9: Threshold $\overline{c}$ along the segment from $(0, 1)$ to $(3, 1)$.

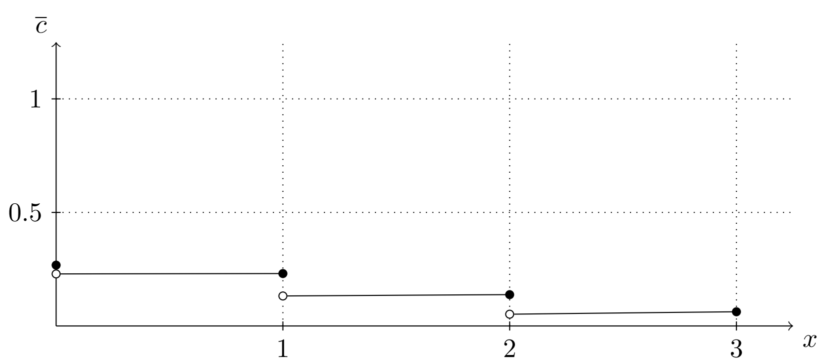


Fig. A.10: Threshold $\overline{c}$ along the segment from $(0, 2)$ to $(3, 2)$.

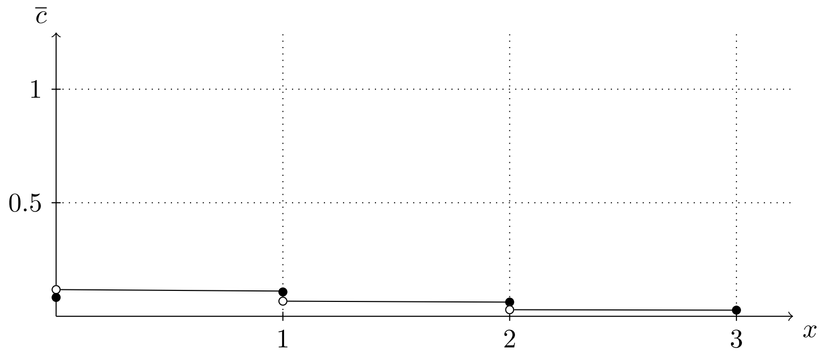
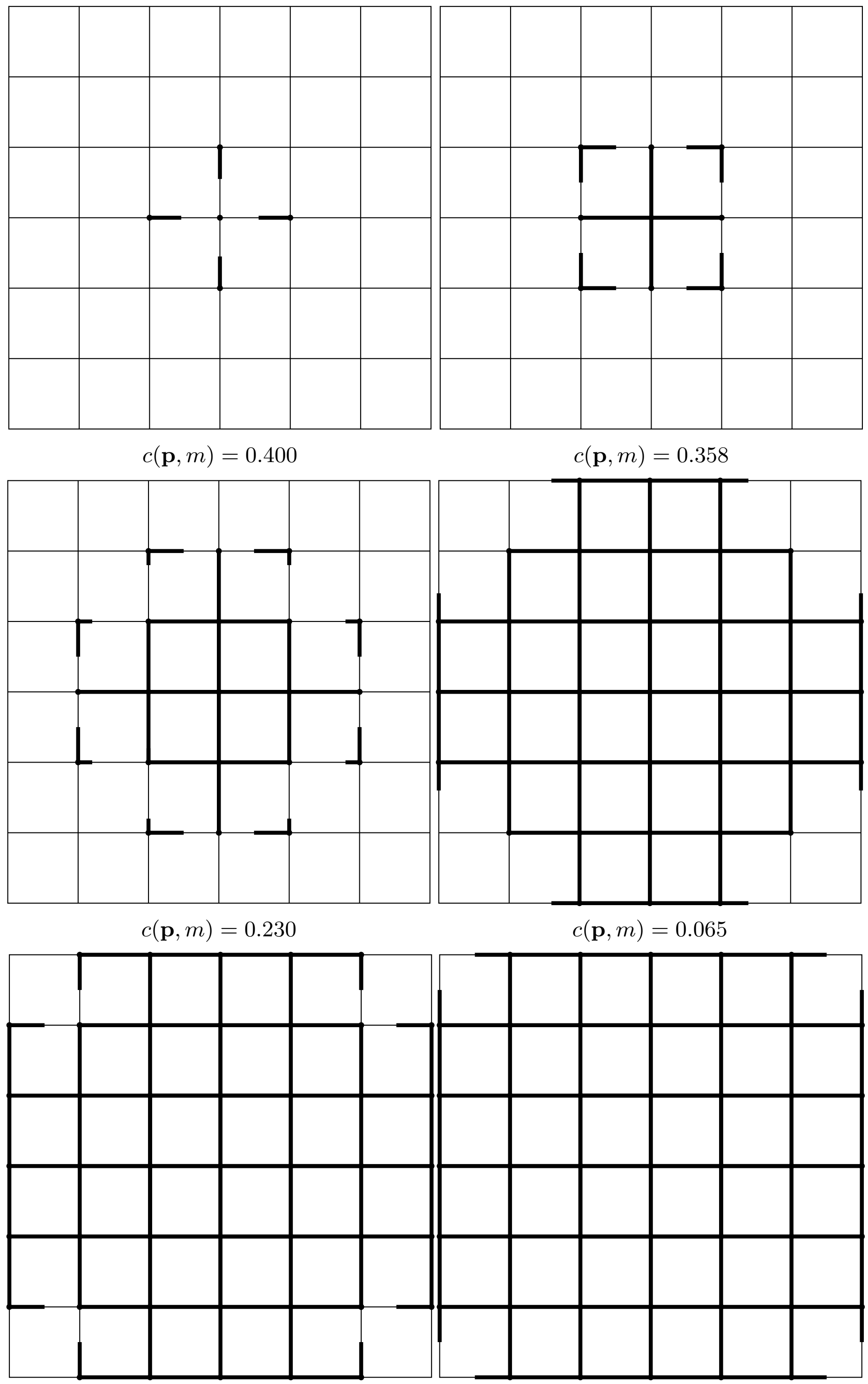


Fig. A.11: Threshold $\overline{c}$ along the segment from $(0,3)$ to $(3,3)$. The jump at the point $(0,3)$ is not to scale—it would be too small to distinguish. The key point is that there is a jump down from the limiting value as x approaches 0 and the value at $(0,3)$.



$c(\mathbf{p}, m) = 0.400$ $c(\mathbf{p}, m) = 0.358$

$c(\mathbf{p}, m) = 0.230$ $c(\mathbf{p}, m) = 0.065$

$c(\mathbf{p}, m) = 0.057$ $c(\mathbf{p}, m) = 0.028$

Fig. A.12: CNE in the case of $M = 6$ for six values of $c(\mathbf{p}, m)$. Not exactly to scale.